

## **ONE DIMENSIONAL CHAOTIC DYNAMICAL SYSTEMS**

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### **Abstract**

In this paper, we studied basic dynamical facts of one-dimensional maps and their dynamical behaviour, especially chaotic behaviour of the maps. Using Lyapunov exponent in this work, it is found that logistic map, tent map, and doubling map are chaotic. We have shown that these maps are topological transitive and sensitive to initial condition. In this work, we have found that one-dimensional maps have complicated dynamical behaviour. We have solved some problems of tent map and after solving those problems, we illustrated that tent map is chaotic. We also discussed the bifurcation of one-dimensional map (logistic map) and sketch bifurcation diagrams by using Mathematica.

### **1. Introduction**

Chaotic dynamical systems constitute a special class of dynamical systems. The study of chaotic phenomena in deterministic dynamical systems has attracted much attention within the last two decades. The use of the term “Chaos” was first introduced into dynamical systems by

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Li and Yorke (Li and Yorke [7]) for a map on a compact interval. Zhou gave a definition of chaos for a topological dynamical system on a general metric space (Zhou [14]) according to the definition of Li and Yorke. Another explicit definition of chaos belongs to Devaney (Devaney [9]). Then Robinson gave a refined definition (Robinson [15]). There are other ways of quantitative measurement of the complicated or chaotic nature of the dynamics. They are Lyapunov exponent, various concepts of (fractal) dimensions, which include box dimension and Hausdorff dimension and topological entropy, respectively.

Chaotic dynamics was made popular by the computer experiments of Robert May and Mitchell Feigenbaum on a mapping known as the logistic map. The remarkable feature of the logistic map is in the simplicity of its form (quadratic) and the complexity of its dynamics. It is the simplest model that shows chaos.

In this paper, we discussed chaotic dynamical behaviour of one dimensional maps, especially logistic map and tent map. The logistic map and the tent map are topologically conjugate, and thus the behaviour of the two maps are in this sense identical under iteration. Depending on the value of parameters, these maps demonstrate a range of dynamical behaviour ranging from predictable to chaotic.

The logistic map  $f(x) = ax(1 - x)$  was recognized as very interesting and representative model of dynamical systems. The simplest logistic maps are quadratic polynomial, which depends on a single parameter “ $a$ ”, but its dynamical behaviour is very complicated. Slight changes in the parameter, “ $a$ ”, of the function can cause the iterated function to change from stable and predictable behaviour to unpredictable behaviour, which is called *chaos* [5]. The behaviour of the logistic function varies greatly as the parameter “ $a$ ” changes. We have shown that logistic map is also chaotic for  $a \geq 4$  with the help of Lyapunov exponent.

The logistic map became a paradigm of chaotic behaviour in the 1970s. In 1964, Lorenz published a paper on the logistic map, essentially noting that even with such a simple equation as the logistic map, non-

periodic behaviour existed. He drew an analogy with the prediction of climate (that is the long term statistical properties of the weather) and concluded that such prediction may not be possible.

By the 1970s, two figures stand out from a number of researchers, who were working on the logistic map or its variants. In the early 1970s, May was investigating the period doubling behaviour of the logistic map, but did not understand what happened beyond the accumulation point of  $\alpha$ ; 3.5699. Actually, the family of logistic maps demonstrates many of these features. The connection between a discrete mapping and the solution of a system of ordinary differential equations in a molecular dynamics simulation is clear, when we realize that the numerical solution of the equations of motion for a system involves an iterative mapping of points in phase space. Although we are solving a problem, which is continuous in time, the differential equation solver transforms this into a discrete time problem. We also discussed chaotic dynamical behaviour of tent map.

## 2. Mathematical Preliminaries

In this section, we have presented some definitions and theorems, which are essential for finding chaotic dynamical behaviour of one dimensional map in the next section.

**Definition 2.1.** If a point  $x$  of a function  $f$  satisfies the condition  $f(x) = x$ , then  $x$  is called *fixed point* of  $f$ , a function has a fixed point at  $c$ , if its graph intersect the line  $y = x$  at  $(c, c)$ .

**Definition 2.2.** A point  $x \in X$  is a *periodic point* [1] of a function  $f$  of period  $n$ , if  $f^n(x) = x$ ,  $n > 1$ . Here  $n$  is the order of  $f$  and the fixed point of  $f^n$  can be written as  $Per_n(f) = fix(f_n)$ .

**Definition 2.3.** A fixed point  $p$  of  $f$  is said to be *hyperbolic*, if and only if  $|f'(p)| \neq 1$ . Also for periodic points  $p$  of order  $n$  of  $f$  is said to be *hyperbolic*, if and only if  $\left| (f^n)'(p) \right| \neq 1$ .

**Definition 2.4.** If  $p$  is a fixed point and  $|f'(p)| < 1$ , then  $p$  is attracting fixed point. If  $p$  is a fixed point and  $|f'(p)| > 1$ , then  $p$  is repelling fixed point. Finally, if  $|f'(p)| = 1$ , then the fixed point  $p$  is called *neutral or indifferent*.

**Definition 2.5.** If  $p$  is a periodic point and  $\left| \left( f^n(p) \right) \right| < 1$ , then  $p$  is attracting periodic point of period  $n$ . If  $p$  is a periodic point and  $\left| \left( f^n(p) \right) \right| > 1$ , then  $p$  is repelling periodic point of period  $n$ .

**Definition 2.6** (Devaney's definition of chaos). Let  $X$  be a compact metric space. A continuous map  $f : X \rightarrow X$  is said to be *chaotic* [1] on  $X$ , if  $f$  satisfies the following properties:

- (i) Periodic points of  $f$  are dense in  $X$ .
- (ii)  $f$  is one-sided topologically transitive.
- (iii)  $f$  has sensitive dependence on initial conditions.

(i) and (ii) of the definition of chaotic map are re-written as follows:

(i)  $per(f)$  is dense in  $X$ .

(ii) For any pair of non-empty sets  $U$  and  $V$  in  $X$ , there exists  $k \in \mathbb{N}$  such that  $f^k(U) \cap V \neq \emptyset$ .

Moreover, the definition of chaotic map is re-written in the term of metric  $d$  on  $X$  as follows:

(i) For any  $x \in X$  and  $\varepsilon > 0$ , we have  $U(x, \varepsilon) \cap per(f) \neq \emptyset$ .

(ii) For any  $x \in X$ ,  $y \in Y$ , and  $\varepsilon > 0$ , there exists  $z \in X$  and  $k \in \mathbb{N}$  such that  $d(x, z) < \varepsilon$  and  $d(f^k(z), y) < \varepsilon$ .

(iii) There exists  $\delta > 0$ , which satisfies that for any  $x \in X$  and any neighbourhood  $N_x$  of  $x$ , there exists  $y \in N_x$  and  $k \in \mathbb{N}$  such that  $d(f^k(x), f^k(y)) \geq \delta$ .

**Definition 2.7** (Chaos in the Li-Yorke sense). A dynamical system  $F : I \rightarrow I$  is continuous and  $I$  is bounded interval. Then  $F$  is chaotic in  $I$  in the Li-Yorke sense [7], if  $F$  has a periodic point in  $I$  of period 3.

**Definition 2.8.** Let  $f$  be a smooth map of the real line. The Lyapunov number  $L(x)$  of the orbit  $\{x_1, x_2, x_3, x_4, \dots\}$  is defined as follows:

$$L(x_1) = \lim_{n \rightarrow \infty} (|f'(x_1)| \dots |f'(x_n)|)^{1/n},$$

if this limit exists. The Lyapunov exponent  $h(x_1)$  is defined as

$$\begin{aligned} h(x_1) &= \lim_{n \rightarrow \infty} \frac{1}{n} [\ln|f'(x_1)| + \dots + \ln|f'(x_n)|] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \ln|f'(x_i)| \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} [\ln \prod_{i=1}^n |f'(x_i)|], \end{aligned}$$

if this limit exists.

**Theorem 2.1.** Suppose  $F : \mathbb{R} \rightarrow \mathbb{R}$  is continuous function and has a periodic point of period three. Then  $F$  has periodic point of all other periods.

**Definition 2.9** (Bifurcation). Let  $f_c$  be a parameterized family of functions. Then there is a bifurcation at  $c_0$ , if there exists  $\varepsilon > 0$  such that whenever  $a$  and  $b$  satisfy  $c_0 - \varepsilon < a < c_0$  and  $c_0 < b < c_0 + \varepsilon$ , then the dynamics of  $f_a$  are different from the dynamics of  $f_b$ . In other words, the dynamics of the function changes when the parameter value crosses through the point  $c_0$ .

**Definition 2.10** (Period doubling bifurcation). A period doubling bifurcation in a dynamical system is a bifurcation in which, the system switches to a new behaviour with twice the period of the original system. Let  $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  be a one parameter family of  $C^3$  maps satisfying

$$(i) f(0, 0) = 0, (ii) \frac{\partial f}{\partial x} \Big|_{\lambda=0, x=0} = -1, (iii) \frac{\partial^2 f}{\partial x^2} \Big|_{\lambda=0, x=0} < 0,$$

(iv)  $\frac{\partial^3 f}{\partial x^3} \Big|_{\lambda=0, x=0} = 0$ , then there are intervals  $(\lambda_1, 0)$ ,  $(\lambda_2, 0)$  and  $\varepsilon > 0$  such that

(a) If  $\lambda \in (\lambda_1, 0)$ , then  $f(\lambda, x)$  has one stable fixed point for  $x \in (-\varepsilon, \varepsilon)$ .

(b) If  $\lambda \in (0, \lambda_2)$ , then  $f(\lambda, x)$  has one unstable fixed points and one stable orbit of period 2 for  $x \in (-\varepsilon, \varepsilon)$ . This type of bifurcation is known as *period-doubling bifurcation*.

The normal form of a period-doubling bifurcation is

$$\frac{dx}{dt} = r - x - x^2.$$

- If  $r < 0$ , there is one equilibrium point, a stable equilibrium point.
- If  $r > 0$ , there are one unstable fixed point and one stable orbit of period 2.

### 3. Chaotic Dynamical Behaviour Using Lyapunov Exponent

**Theorem 3.1.** *If at least one of the average Lyapunov exponents is positive, then the system is chaotic; if the average Lyapunov exponent is negative, then the orbit is periodic; and when the average Lyapunov exponent is zero, a bifurcation occurs.*

**Problem 3.1.** Show that the logistic map  $f(x) = 4x(1 - x)$  is chaotic.

**Solution.** Let  $\{x_1, x_2, x_3, x_4, \dots, x_k\}$  be a periodic orbit of the logistic map  $f$ . Then the stability of  $f$  is determined by the derivative of  $f^k$ . So by the chain rule along a cycle, we have

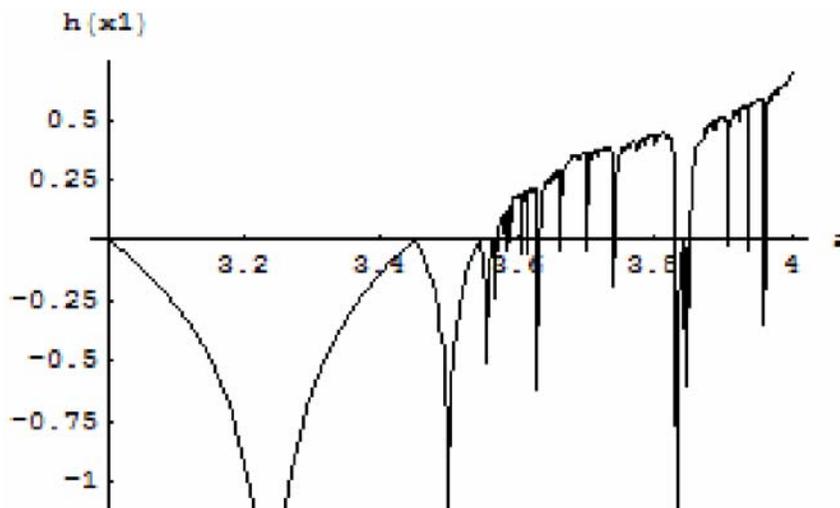
$$(f^k)'(x_1) = f'(x_1) \cdot f'(x_2) \cdot \dots \cdot f'(x_k).$$

The derivatives of  $f$  on  $[0, 1]$  range between 0 and 4 (in magnitude), there is no priori reason to expect the product  $k$  of these numbers to have a simple expression or to have magnitude greater than one. In fact, the product amounts to precisely  $2^k$  for a period  $k$ -orbit. Hence all periodic points of  $f$  are sources that is the orbit is not asymptotically periodic.

Now, we compute the Lyapunov exponent  $h(x_1)$

$$\begin{aligned}
 h(x_1) &= \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=1}^k \ln|f'(x_i)| \\
 &= \ln 2 > 0.
 \end{aligned}$$

Thus  $f(x)$  is chaotic on  $[0, 1]$ .



**Figure 3.1.** Lyapunov exponent of logistic map.

The graph shows the Lyapunov exponent

$h(x_1) = \lim_{n \rightarrow \infty} \frac{1}{n} [\ln(|f'(x_1)|) + \dots + \ln|f'(x_n)|]^{1/n}$  of the logistic map for  $a = 4$ . The graph of the function  $h(x_1)$  is plotted for values of  $a$  between 3 and 4. A positive Lyapunov exponent indicates chaotic behaviour.

**Problem 3.2.** The tent map  $T(x) = \begin{cases} 2x, & 0 \leq x \leq \frac{1}{2} \\ 2(1-x), & \frac{1}{2} \leq x \leq 1 \end{cases}$  is

chaotic on  $[0, 1]$ .

**Solution.** In this case,  $|T'(x)| = 2 > 1$ . So, the tent map is not attracted to a sink and then is not asymptotically periodic. It is easy to compute the Lyapunov exponent as

$$\begin{aligned} h(x_1) &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \ln|T'(x_i)| \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \ln 2 \\ &= \ln 2 > 0. \end{aligned}$$

Since each orbit of  $T(x)$  is asymptotically periodic and Lyapunov exponent  $h(x_1) > 0$ , the tent map is chaotic on  $[0, 1]$ .

#### 4. Chaotic Dynamical Behaviour of the One Dimensional Maps

##### 4.1. Logistic map

The quadratic map (logistic map) is defined by the equation  $x_{n+1} = f_a(x_n) = ax_n(1 - x_n)$ , where  $x_n$  is a number between zero and one, and represents the population at year  $n$ , and hence  $x_0$  represents the initial population (at year 0) and  $a$  is a positive number, and represents a combined rate for reproduction and starvation.

In the last section, we have discussed about fixed point and periodic point and that time we see that a fixed point is a periodic point of period one. We will consider the quadratic map  $f_a(x)$  on the interval  $0 < x < 1$ , as a function of the parameter  $a$ . With  $a$  between 0 and 1, the population will eventually die, independent of the initial population. With  $a$  between

1 and 2, the population will quickly approach the value  $1 - \frac{1}{a}$ , independent of the initial population. With  $a$  between 2 and 3, the population will also eventually approach the same value  $1 - \frac{1}{a}$ , but first will fluctuate around that value for some time. The rate of convergence is linear, except for  $a = 3$ , when it is dramatically slow, less than linear.

For  $a > 3$ , changes to repelling and a 2-cycle is born.

The system exhibits some interesting phenomena [4], which cannot be observed from the continuous logistic system.

For  $3 < a < 1 + \sqrt{6} \approx 3.45$ , the two cycle is stable the population may oscillate between two values forever.

For  $a = 1 + \sqrt{6} \approx 3.544090$ , the 2-cycle becomes unstable and a stable 4-cycle is born, then the population may oscillate between four values forever.

For  $1 + \sqrt{6} < a < 1 + 2\sqrt{2} \approx 3.544090$ , the 4-cycle becomes unstable and a stable 8-cycle is born, when  $a$  is slightly bigger than 3.54, then 8 than 16, 32 etc..

In general, a stable  $2^k$  cycle is born at  $a_k$  and becomes unstable at  $a_{k+1}$ , where

$$\begin{array}{ll} \alpha_1 = 3 & \alpha_5 = 3.568759... \\ \alpha_2 = 3.449... & \alpha_6 = 3.569692... \\ \alpha_3 = 3.54409... & \alpha_7 = 3.5698... \\ \alpha_4 = 3.5644... & \alpha_8 = 3.569946.... \end{array}$$

It should be clear that as  $n$  goes large, these values are approaching a limit

$$\alpha_\infty = \lim_{n \rightarrow \infty} \alpha_n = 3.569946....$$

It means that an  $\infty$  cycle for the value of  $a$ . Note that the successive bifurcations come faster and faster. The convergence is essentially geometric; the limit of large  $n$ , the distance between successive transitions shrinks by a constant factor

$$\delta = \lim_{n \rightarrow \infty} \frac{a_n - a_{n-1}}{a_{n+1} - a_n} = 4.6692016091 \dots$$

At  $a$  approximately 3.57 is the onset of chaos, at the end of the period-doubling cascade. Most values beyond 3.57 exhibit chaotic behaviour, but there are still certain isolated ranges of  $a$  that show non-chaotic behaviour; these are sometimes called *islands of stability*. For instance, beginning at 3.83, there is a range of parameters  $a$ , which show oscillation among three values, and for slightly higher values of  $a$  oscillation among 6 values, then 12 etc..

The development of the chaotic behaviour of the logistic sequence as the parameter  $a$  varies from approximately 3.5699 to approximately 3.8284 is sometimes called the Pomeau-Manneville scenario, which is characterized by a period phase interrupted by bursts of a periodic behaviour.

#### 4.2. Tent map

$$\text{Define as iterative map by } x_{n+1} = T_b(x_n) = \begin{cases} bx_n & \text{for } x_n < 1/2, \\ b(1 - x_n) & \text{for } 1/2 \leq x_n, \end{cases}$$

where  $b$  is a positive real constant, and  $x_n \in [0, 1]$  is called *tent map*. Although the form of the tent map is simple and the equations involved are linear, for certain parameter values, this system can display highly complex behaviour and even chaotic phenomena. For certain parameter values, the mapping undergoes stretching and folding transformations and display sensitivity to initial conditions and periodicity.

By varying the parameter  $b$ , the following behaviour is observed:

(i) If  $b$  is less than 1, the point  $x = 0$  is an attractive fixed point of the system for all initial values of  $x$ .

(ii) If  $b$  is 1, all values of  $x$  less than or equal to  $\frac{1}{2}$  are fixed points of the system.

(iii) If  $b$  is greater than 1, the system has two fixed points, are at 0 and the other at  $\frac{b}{b+1}$ . Both fixed points are unstable. For example, when  $b$  is 1.5, there is a fixed point at  $x = 0.6$  (because  $1.5(1-0.6) = 0.6$ , but starting at  $x = 0.61$ , we get  $0.61 \rightarrow 0.585 \rightarrow 0.6225 \rightarrow 0.56625 \rightarrow 0.650625 \dots$ ).

(iv) If  $b$  is between 1 and the square root of 2, the system maps a set of intervals between  $b - b^2 / 2$  and  $b / 2$  to themselves. This set of intervals is the Julia set of the map that is it is the smallest invariant subset of the real line under the map. If  $b$  is greater than the square root of 2, these interval merge, as the Julia set is the whole interval from  $b - b^2 / 2$  to  $b / 2$ .

(v) If  $b$  is between 1 and 2, the interval  $[b - b^2 / 2, b / 2]$  contains both periodic and non-periodic points, although all of the orbits are unstable that is nearby points move away from the orbits rather than towards them.

(vi) If  $b$  equals 2, the system maps the interval  $[0, 1]$  onto itself. There are new periodic points with every orbit length within this interval, as well as non-periodic points. The periodic points are dense in  $[0, 1]$ , so the map has become chaotic.

(vii) If  $b$  is greater than 2, the map's Julia set becomes disconnected, and breaks up into a cantor set within the interval  $[0, 1]$ . The Julia set still contain an infinite number of both non-periodic and periodic points, but almost every point within  $[0, 1]$  will now eventually diverges towards infinity. The canonical cantor set is the Julia set of the tent map for  $b = 3$ .

**Problem 4.1.** Iterate the tent function numerically and graphically for the following  $b$  and  $x_0$  values:

$$(I) \ b = \frac{1}{2} : (i) \ x_0 = \frac{1}{4}, (ii) \ x_0 = \frac{1}{2}, (iii) \ x_0 = \frac{3}{4}.$$

$$(II) \ b = 1 : (i) \ x_0 = \frac{1}{3}, (ii) \ x_0 = \frac{2}{3}.$$

$$(III) \ b = \frac{3}{2} : (i) \ x_0 = \frac{3}{5}, (ii) \ x_0 = \frac{6}{13}, (iii) \ x_0 = \frac{1}{3}.$$

$$(IV) \ b = 2 : (i) \ x_0 = \frac{1}{3}, (ii) \ x_0 = \frac{1}{5}, (iii) \ x_0 = \frac{1}{7}, (iv) \ x_0 = \frac{1}{11}.$$

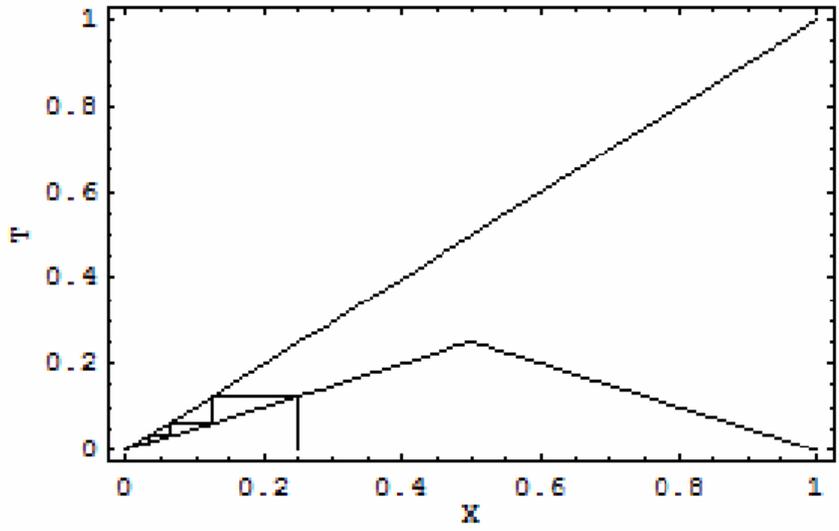
**Solution.** For simplicity, the iterates will be listed as  $\{x_0, x_1, x_2, \dots, x_n\}$ . The solutions are as follows:

$$(I) \ b = \frac{1}{2}$$

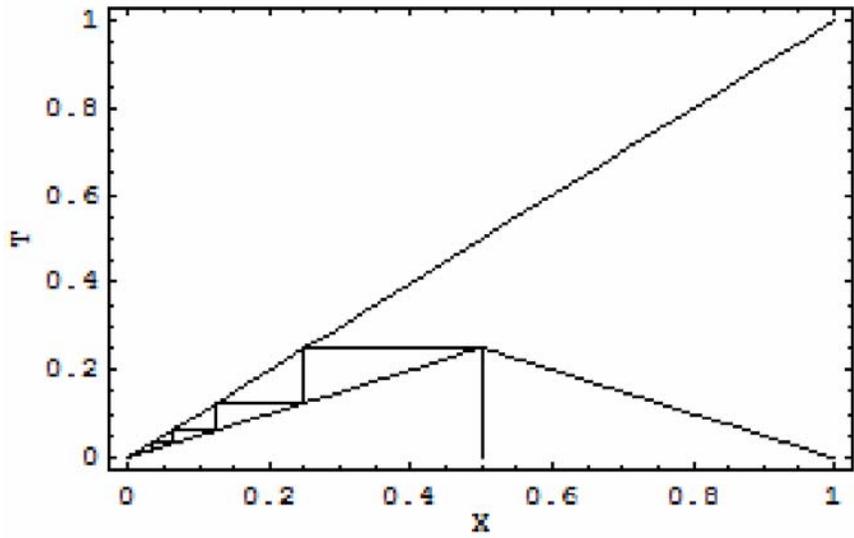
$$(i) \ \left\{ \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots, \frac{1}{4 \times 2^n} \right\};$$

$$(ii) \ \left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots, \frac{1}{2^{n+1}} \right\};$$

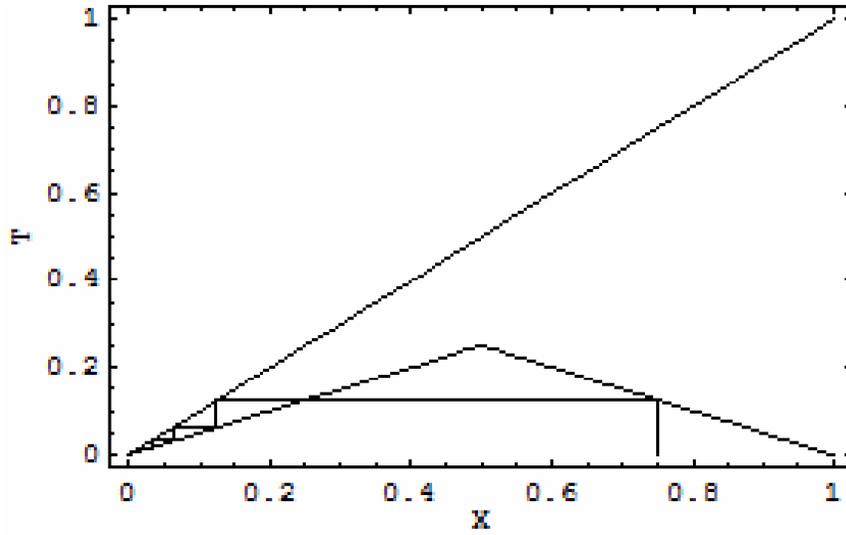
$$(iii) \ \left\{ \frac{3}{4}, \frac{3}{8}, \frac{3}{16}, \dots, \frac{3}{4 \times 2^n} \right\}. \text{ In each case, } x_n \rightarrow 0 \text{ as } n \rightarrow \infty.$$



(a)



(b)



(c)

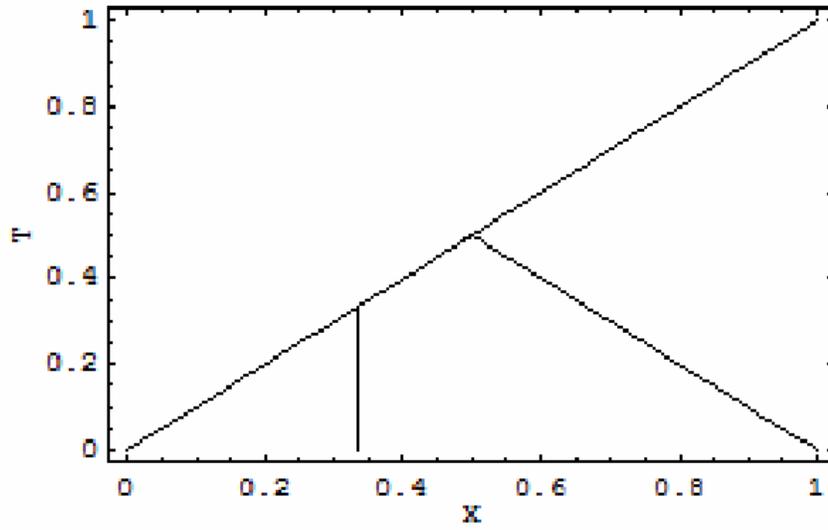
**Figure 4.1.** Graphical iterations when  $b = \frac{1}{2}$ : (a)  $x_0 = \frac{1}{4}$ ; (b)  $x_0 = \frac{1}{2}$ ; and (c)  $x_0 = \frac{3}{4}$ .

(II)  $b = 1$

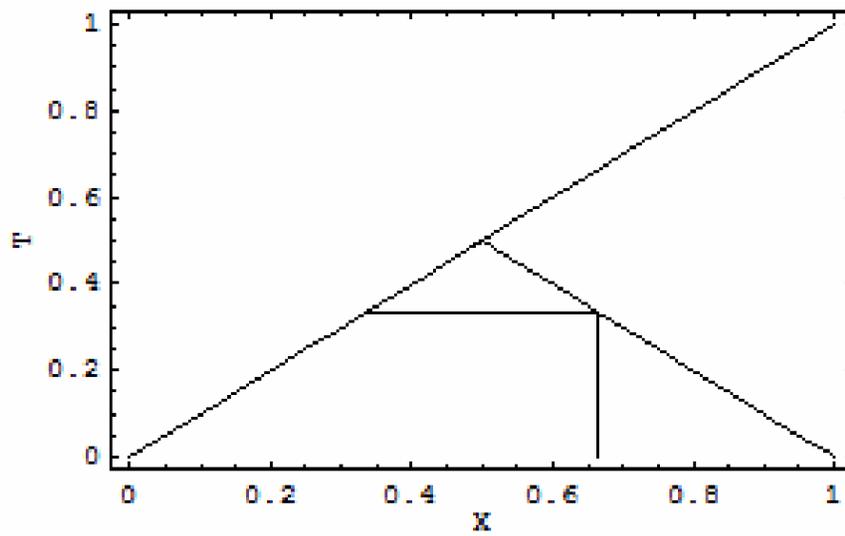
(i)  $\left\{ \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \dots, \frac{1}{3}, \dots \right\}$ ;

(ii)  $\left\{ \frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \dots, \frac{1}{3}, \dots \right\}$ .

The orbits tend to points of period one in the range  $\left[0, \frac{1}{2}\right]$ .



(a)



(b)

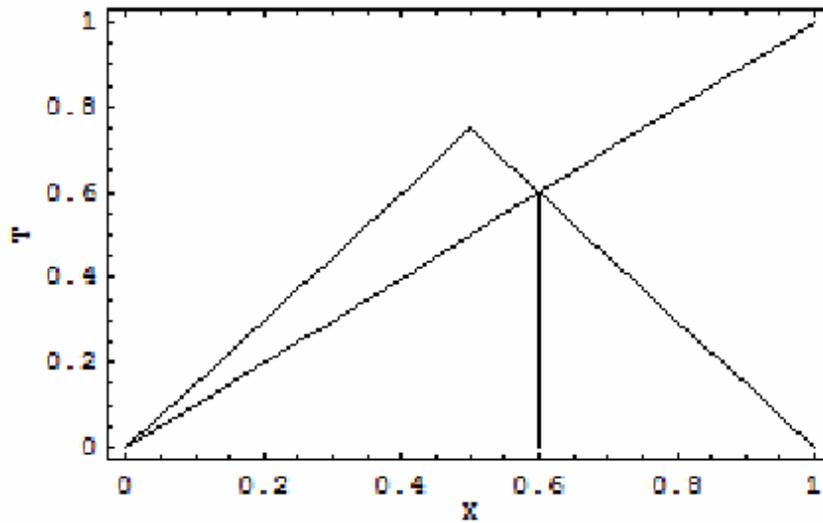
**Figure 4.2.** Graphical iteration when  $b = 1$  : (a)  $x_0 = \frac{1}{3}$  and (b)  $x_0 = \frac{2}{3}$ .

$$(III) b = \frac{3}{2}$$

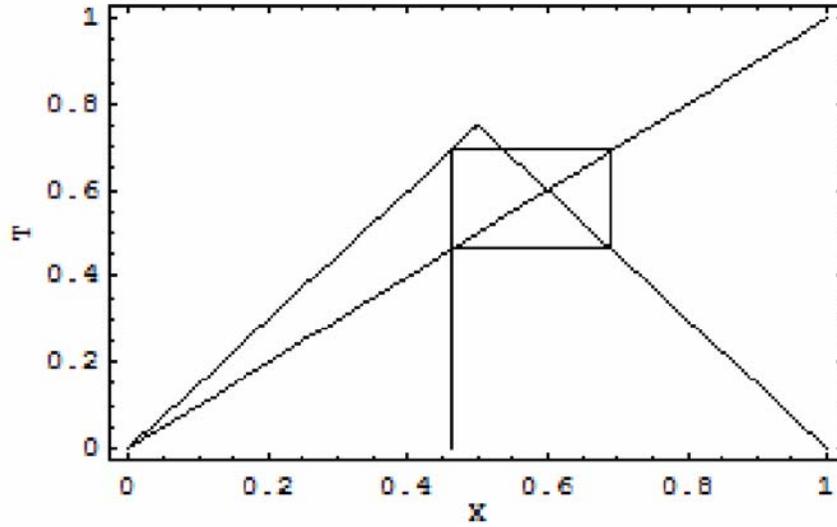
$$(i) \left\{ \frac{3}{5}, \frac{3}{5}, \frac{3}{5}, \dots, \frac{3}{5}, \dots \right\};$$

$$(ii) \left\{ \frac{6}{13}, \frac{9}{13}, \frac{6}{13}, \frac{9}{13}, \dots, \frac{6}{13}, \frac{9}{13}, \dots \right\};$$

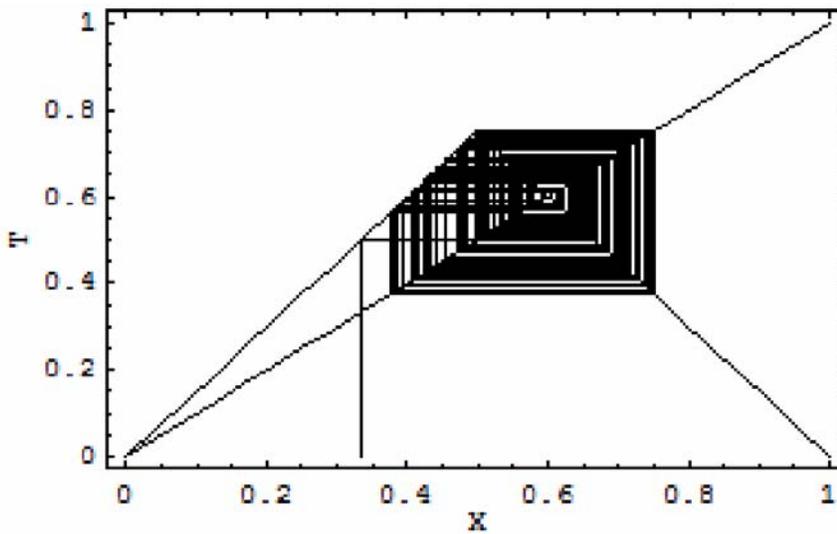
$$(iii) \left\{ \frac{1}{3}, \frac{1}{2}, \frac{3}{4}, \frac{8}{3}, \frac{9}{16}, \frac{21}{32}, \frac{33}{64}, \frac{93}{128}, \frac{105}{256}, \frac{315}{512}, \frac{591}{1024}, \dots \right\}.$$



(a)



(b)



(c)

**Figure 4.3.** Graphical iterations when  $b = \frac{3}{2}$  : (a)  $x_0 = \frac{3}{5}$ ; (b)  $x_0 = \frac{6}{13}$ ; and (c)  $x_0 = \frac{1}{3}$ , for 200 iterations.

In case (i), the iterate  $x_{n+1}$  is equal to  $x_n$  for all  $n$ . This type of sequence displays period-one behaviour. In case (ii), the iterate  $x_{n+2}$  is equal to  $x_n$  for all  $n$ , and the result is period-two behaviour. In case (iii), the first 11 iterations are listed, but other methods need to be used in order to establish the long term behaviour of the sequence.

(IV)  $b = 2$

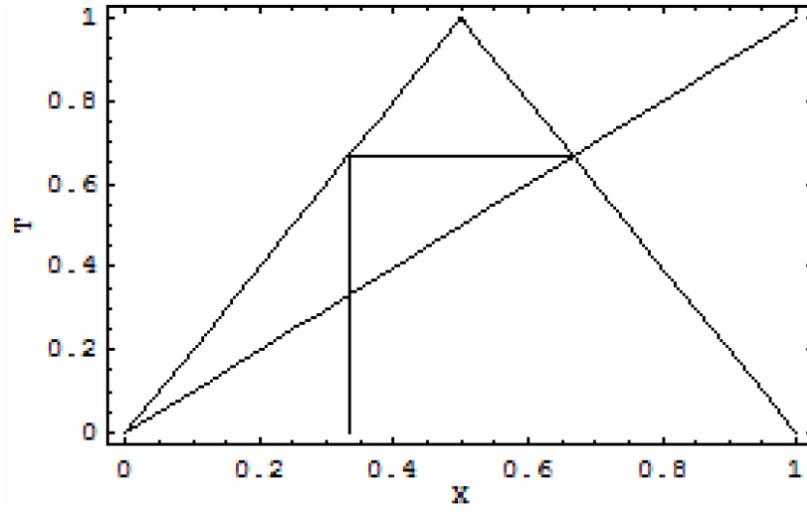
$$(i) \left\{ \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \dots, \frac{2}{3}, \dots \right\};$$

$$(ii) \left\{ \frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \frac{2}{5}, \frac{4}{5}, \dots, \frac{2}{5}, \frac{4}{5}, \dots \right\};$$

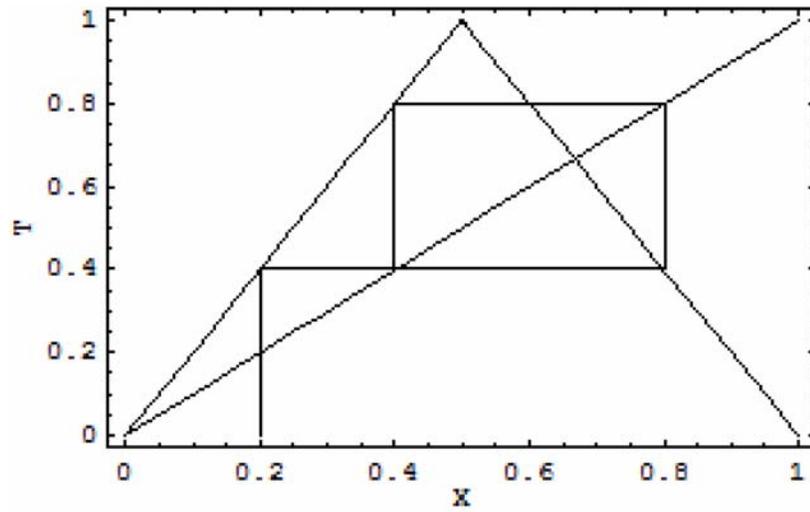
$$(iii) \left\{ \frac{1}{7}, \frac{2}{7}, \frac{4}{7}, \frac{6}{7}, \frac{2}{7}, \frac{4}{7}, \frac{6}{7}, \dots, \frac{2}{7}, \frac{4}{7}, \frac{6}{7}, \dots \right\};$$

$$(iv) \left\{ \frac{1}{11}, \frac{2}{11}, \frac{4}{11}, \frac{8}{11}, \frac{6}{11}, \frac{10}{11}, \frac{2}{11}, \dots, \frac{2}{11}, \frac{4}{11}, \frac{8}{11}, \frac{6}{11}, \frac{10}{11}, \dots \right\}.$$

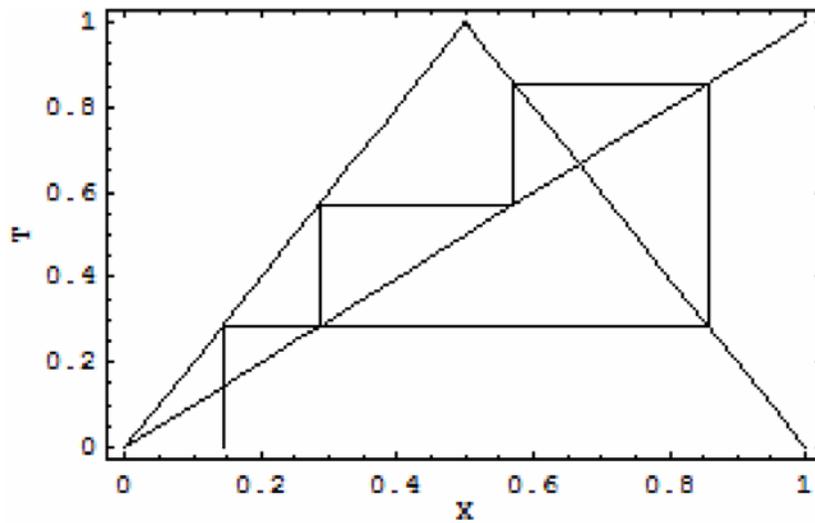
The sequences behave as follows: (i) there is period-one behaviour, (ii) there is period-two behaviour, (iii) there is a period-three sequence, and (iv) there is a period-five sequence.



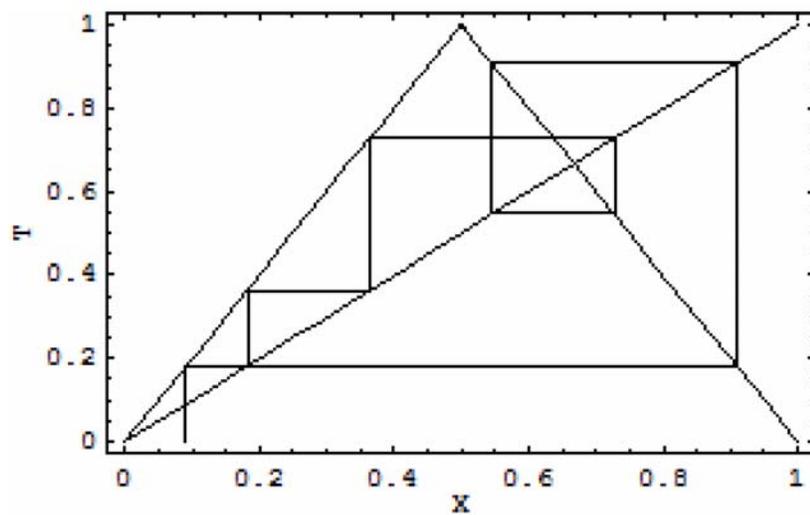
(a)



(b)



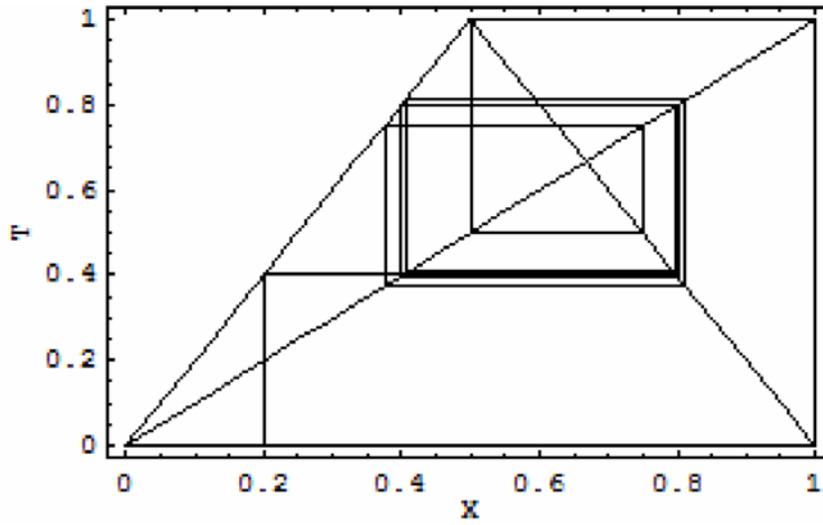
(c)



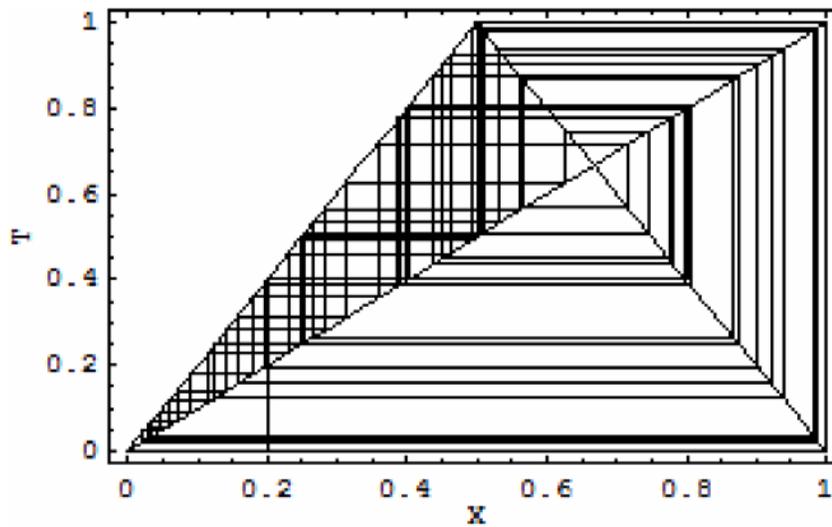
(d)

**Figure 4.4.** Graphical iterations when  $b = 2$  : (a)  $x_0 = \frac{1}{3}$ ; (b)  $x_0 = \frac{1}{5}$ ; (c)  $x_0 = \frac{1}{7}$ ; and (d)  $x_0 = \frac{1}{11}$ , for 200 iterations.

Again, if we iterate the tent function graphically for  $b = 2$ : (a)  $x_0 = 0.2$  and (b)  $x_0 = 0.2001$ , we can get the following figures:



(a)



(b)

**Figure 4.5.** Graphical iterations when  $b = 2$ : (a)  $x_0 = 0.2$  and (b)  $x_0 = 0.2001$ , for 200 iterations.

Each of the diagrams (Figures 4.1-4.5) can be reproduced by using Mathematica. When  $b = \frac{3}{2}$ , the tent map displays sensitivity to initial conditions and can be described as being chaotic [12]. The iterative path plotted in Figure 4.3(c) appears to wander randomly. It is still not clear whether the path is chaotic or whether the path is periodic of a very high period. Figure 4.5 clearly shows the sensitivity to initial conditions. Again, it is not clear in case (ii) whether the path is chaotic or of a very high period. It is clear from the diagrams is that the three basic properties of chaos and they are mixing, periodicity, sensitivity to initial conditions, and they are all exhibited for certain parameter values. Indeed, a now-famous result due to Li and Yorke [7] states that if a system displays period-three behaviour, then the system can display periodic behaviour of any period. Li and Yorke then go on to prove that the system can display chaotic phenomena. Hence when  $b = 2$ , system  $x_{n+1} = T_b(x_n)$  is chaotic since it has a period-three sequence (Problem 4.1 (iv)(iii)).

### 5. Period-Doubling of the Logistic Map

The logistic map is defined by

$$f_a(x) = ax(1-x), x \in [0, 1], a > 0 \text{ for given } x \in [0, 1].$$

By varying the parameter  $a$ , the following behaviour is observed:

(i)  $0 < a < 1$  :

In this case,  $f_a(x)$  gives us a single fixed point  $x = 0$  is attracting (or stable).

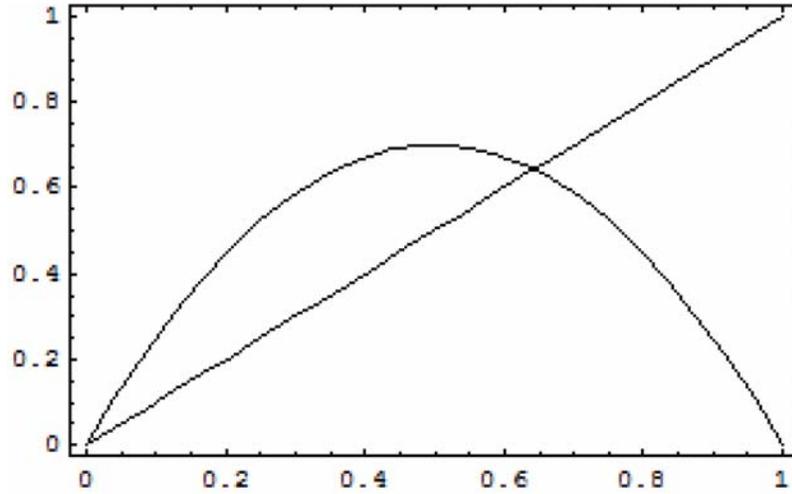
(ii)  $1 < a < 3$  :

In this case, solving  $f_a(x) = x$  gives us two fixed points  $x = 0$  and  $x = 1 - 1/a$ .

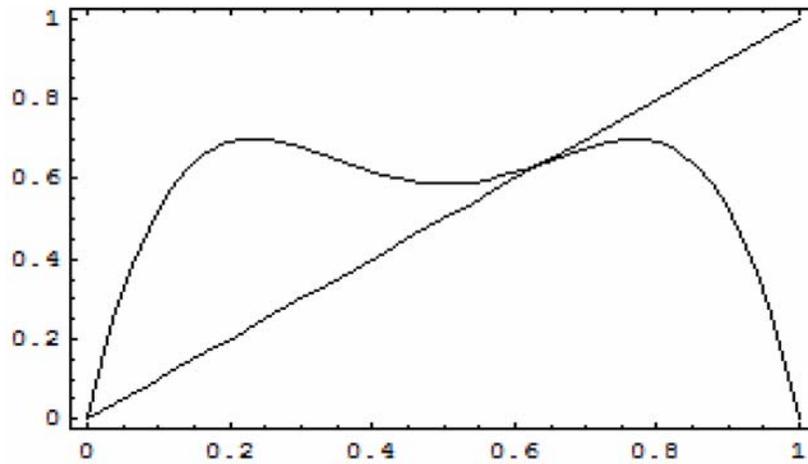
$$f'_a(0) = a > 1 \Rightarrow x = 0 \text{ is repelling (or unstable).}$$

$$|f'_a(1 - 1/a)| = |2 - a| < 1 \Rightarrow x = 1 - 1/a \text{ is attracting (or stable).}$$

Then, we can show that all the pre-images of this interval are  $(0, 1)$ .



**Figure 5.1.**  $f_a(x)$  for  $a = 2.8$ .



**Figure 5.2.**  $f_a^2(x)$  for  $a = 2.8$ .

(iii)  $a > 3$  :

In this case, both of the two fixed points are repelling. But the periodic points appear 2-cycle.

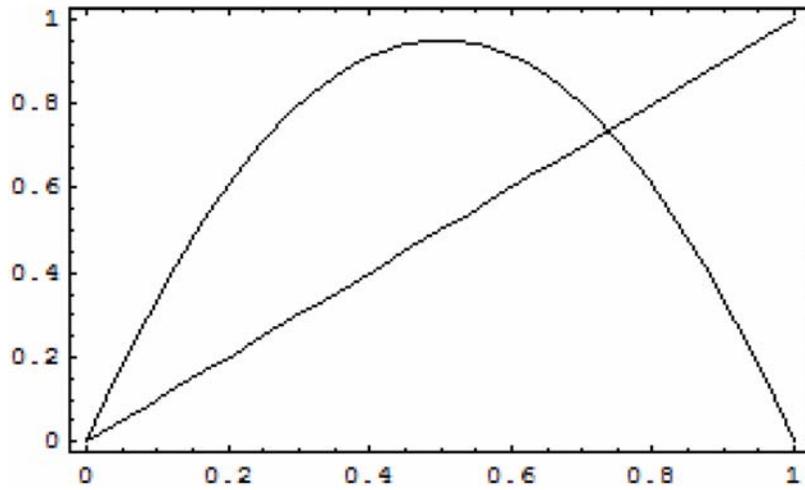
Solving  $f_a^2(x) = x$  gives us four roots  $x = 0, 1 - 1/a, \frac{a+1 \pm \sqrt{(a-3)(a+1)}}{2a}$ .

The first two are clearly the repelling fixed points. The second two are two-periodic points. We have  $(f_a^2)'(x_0) = f_a'(x_1)f_a'(x_0)$ , where  $x_1 = f_a(x_0)$ .

For  $x_0 = \frac{a+1 \pm \sqrt{(a-3)(a+1)}}{2a}$ ,  $(f_a^2)'(x_0) < 1$ .

Therefore, the 2-cycle is attracting for  $|4 + 2a - a^2| < 1$ .

That is  $3 < a < 1 + \sqrt{6} \approx 3.449\dots$



**Figure 5.3.**  $f_a(x)$  for  $a = 3.8$ .

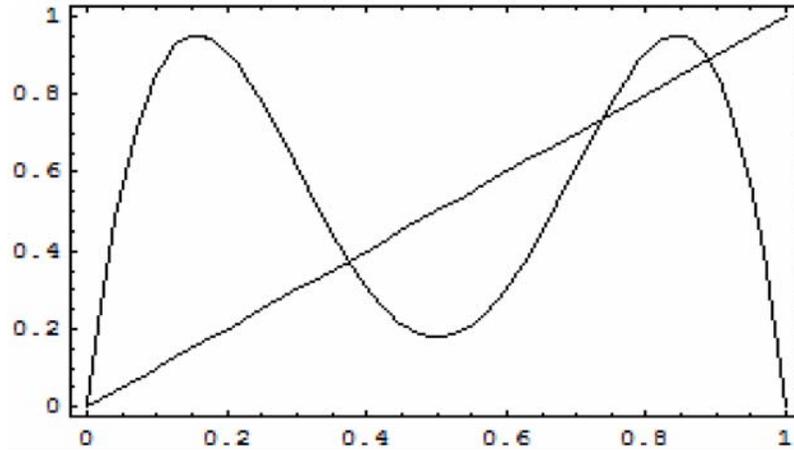


Figure 5.4.  $f_a^2(x)$  for  $a = 3.8$ .

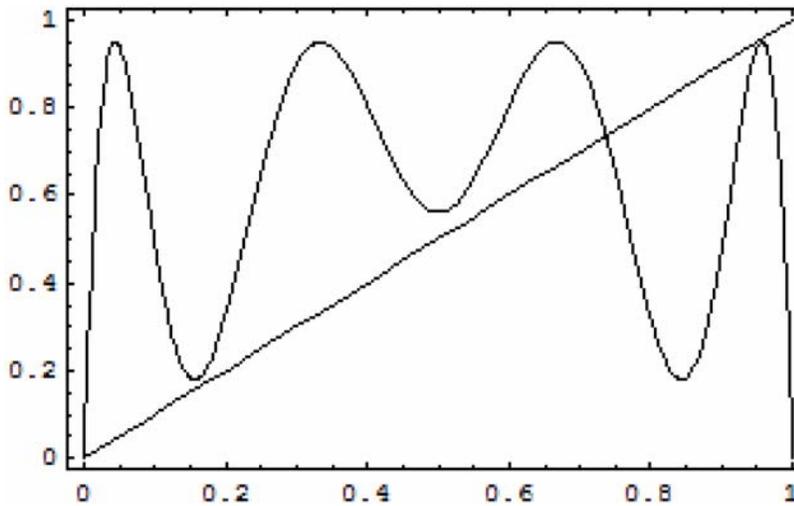


Figure 5.5.  $f_a^3(x)$  for  $a = 3.8$ .

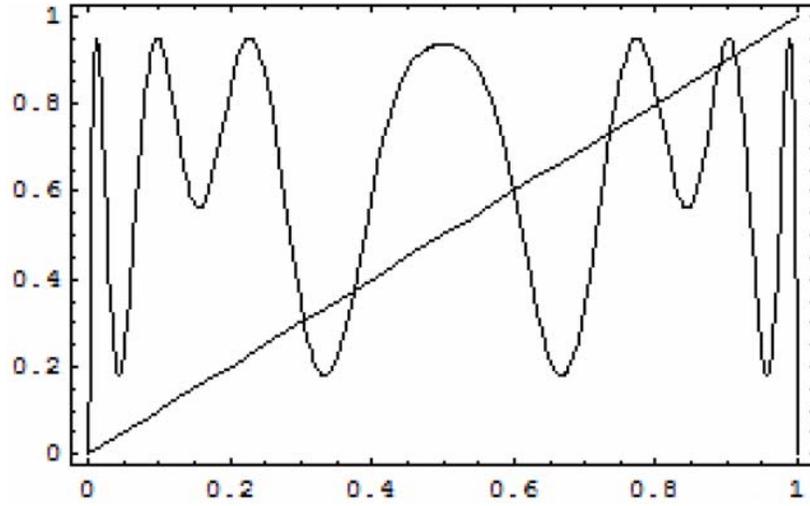


Figure 5.6.  $f_a^4(x)$  for  $a = 3.8$ .

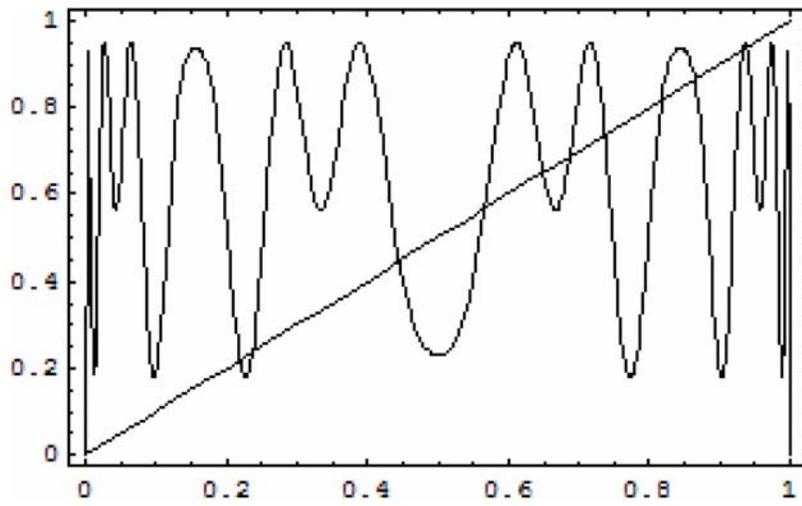


Figure 5.7.  $f_a^5(x)$  for  $a = 3.8$ .

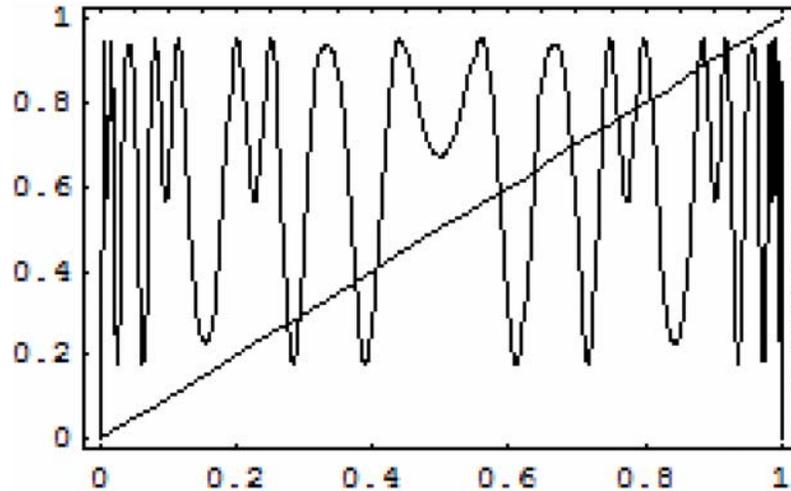


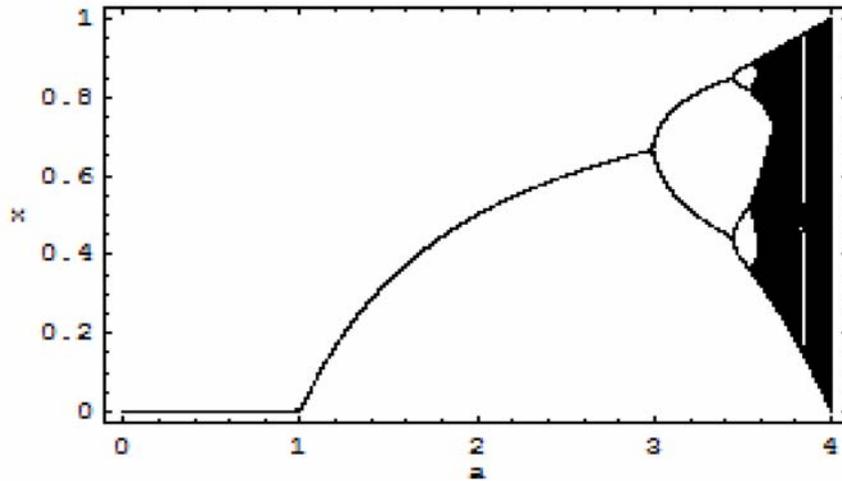
Figure 5.8.  $f_a^6(x)$  for  $a = 3.8$ .

Every point is eventually attracted to this 2-cycle. Numerically, one usually finds that every point in  $[0, 1]$  is attracted to this 2-cycle because both 0 and  $1 - \frac{1}{a}$  are repelling, when  $a > 1 + \sqrt{6}$  the 2-cycle is repelling, but a 4-cycle appears  $4 > a > 1 + \sqrt{6} \approx 3.449$ .

When  $3.449... < a < 3.54409...$ , there is an attracting 4-cycle. Numerically, every point is eventually attracted to it.  $3.54409... < a < 3.5644...$ , there is an attracting 8-cycle, which numerically attracts every point in  $(0, 1)$ .  $3.5644... < a < 3.568759...$ , there is an attracting 16-cycle. This is the so-called period doubling bifurcation. Note that the successive bifurcations come faster and faster. Ultimately, the intervals of  $2n$ -cycles converge to a point  $a_\infty \approx 3.569946...$  as  $n \rightarrow \infty$ . It is relatively easy to show that the logistic map is chaotic on an invariant Cantor set for  $a > 2 + \sqrt{5} \approx 4.236$  (Devaney [9], pp. 31-50; Gulick [16], pp. 112-126; Holmgren [3], pp. 69-85), but in fact, it is also chaotic for all  $a > 4$  (Robinson [15], pp. 33-37; Kraft [17]).

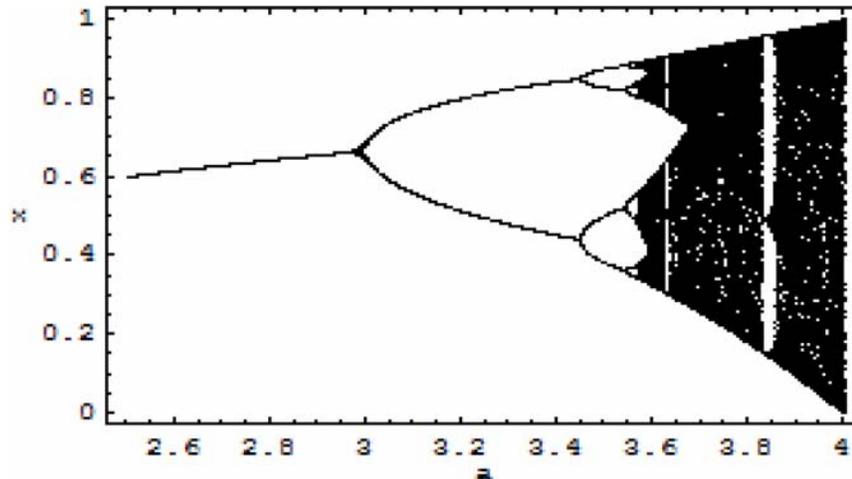
### 6. Bifurcation Diagram of Logistic Map

A bifurcation diagram is a visual summary of the succession of period-doubling produced as  $a$  increases. The next figure shows the bifurcation diagram of the logistic map,  $a$  along the  $x$ -axis. For each value of the system is first allowed to settle down and then the successive values of  $x$  are plotted for a few hundred iterations.



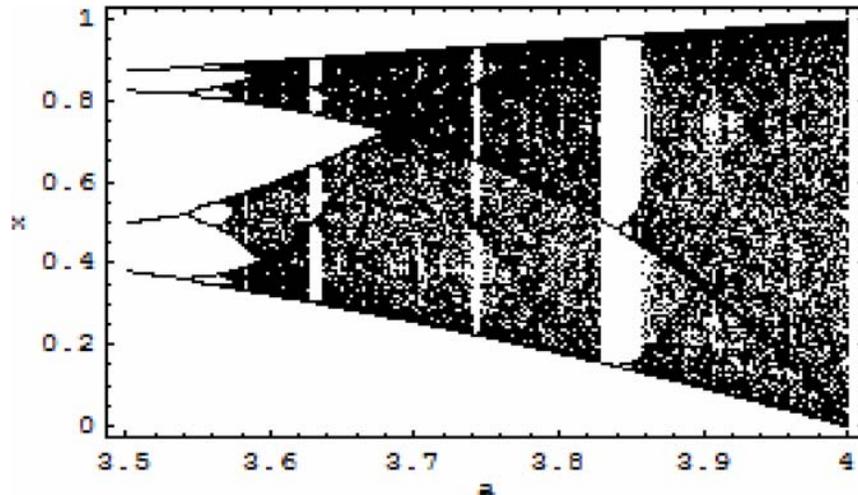
**Figure 6.1.** Bifurcation diagram  $a$  between 0 and 4.

We observe that for  $a$  less than one, all the points are plotted at zero. Zero is the one point attractor for  $a$  less than one. For  $a$  between 1 and 3, we still have one-point attractors, but the 'attracted' value of  $x$  increases as  $a$  increases, at least to  $a = 3$ . Bifurcations occur at  $a = 3$ ,  $a = 3.45$ ,  $3.54$ ,  $3.564$ ,  $3.569$  (approximately), etc., until just beyond  $3.57$ , where the system is chaotic. However, the system is not chaotic for all values of  $a$  greater than  $3.57$ .



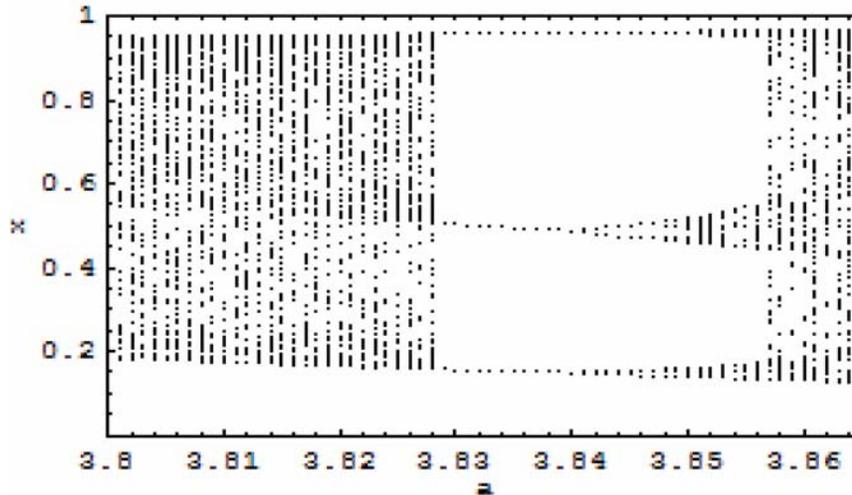
**Figure 6.2.** Bifurcation diagram  $a$  between 2.4 and 4.

From the above figure, we can see that we are getting nice clean bifurcation, and we can see some nice details in the chaotic parts of the diagram. The bifurcation diagram is a fractal because if we zoom in on the above mentioned value  $a = 3.82$  and focus on one arm of the three, the situation nearby looks like a shrunk and slightly distorted version of the whole diagram. The same is true for all other non-chaotic points. This is an example of the connection between chaos and fractals. Let us zoom in a bit.



**Figure 6.3.** Bifurcation diagram  $a$  between 3.5 and 4.

Here, we can see some new lines appear. For the non chaotic parts of the diagram, these lines trace the values that  $x$  visits before settling into an oscillation. The windows of period three (at about  $a = 3.83$ ), period five (at about  $a = 3.74$ ), and period six (at about  $a = 3.63$ ) are clearly visible in the above diagram. Notice that at several values of  $a$ , greater than 3.57, a small number of  $x$  values are visited. These regions produce the 'white space' in the diagram. In fact, between 3.57 and 4, there is a rich interleaving of chaos and order. A small change in  $a$  can make a stable system chaotic, and vice versa.



**Figure 6.4.** Bifurcation diagram  $a$  between 3.8 and 3.865.

Look closely at  $a = 3.83$  and we will see a three-point attractor.

The map needs to be redefined for  $a > 4$  because values  $x > 1$  of can occur, which ultimately divergent. Also, slope outside  $0 \leq x \leq 1$  has magnitude strictly greater than 1. So, we restrict the control parameter  $a$  to the range  $0 \leq a \leq 4$ . So that the logistic map maps the interval  $0 \leq x \leq 1$  into itself. The behaviour is much less interesting for other values of  $x$  and  $a$ .

## 7. Conclusion

The study of low dimensional dynamical systems, which exhibit chaos is a very active area of current research. A very useful introductory account can be found in Schuster [18]. It was long thought that the complex behaviour of systems of many degrees of freedom was inherently different to that of simple mechanical systems. It is now known that simple one-dimensional nonlinear system can indeed show very complex behaviour.

In this paper, we see that the equation of the logistic map is not particularly complex, so we might not expect anything exciting to happen. For values of  $a$  less than about 3.45, the value of  $x$  will bounce around a few times and quickly converge on a value. However, if we keep increasing the value of  $a$ , things become more interesting. As  $a$  continues to increase  $x$  will no longer settle to a single value, but instead oscillate between two values. If we keep increasing the value of  $a$ , we will find that  $x$  now starts to oscillate between 4 values, and then 8, then 16 and so on. However, if we keep increasing  $a$ , we will occasionally see that some values of  $a$  again seem to show more ordered behaviour.

The relative simplicity of the logistic map makes it an excellent point of entry into a consideration of the concept of chaos. In this paper, we observe that a rough description of chaos is that chaotic systems exhibit a great sensitivity to initial conditions -- a property of the logistic map for most values of  $a$  between about 3.57 and 4 (as noted above). A common source of such sensitivity to initial conditions is that the map represents a repeated folding and stretching of the space on which it is defined. In the case of the logistic map, the quadratic difference equation describing it may be thought of as a stretching-and-folding operation on the interval  $(0, 1)$ . On the other hand, when  $b = 2$ , the tent map has become chaotic.

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